MIRROR COUPLING OF REFLECTING BROWNIAN MOTION AND AN APPLICATION TO CHAVEL'S CONJECTURE

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Abstract. In a series of papers, Burdzy et. al. introduced the *mirror coupling* of reflecting Brownian motions in a smooth bounded domain $D \subset \mathbb{R}^d$, and used it to prove certain properties of eigenvalues and eigenfunctions of the Neumann Laplaceian on D.

In the present paper we show that the construction of the mirror coupling can be extended to the case when the two Brownian motions live in different domains $D_1, D_2 \subset \mathbb{R}^d$.

As an application of the construction, we derive a unifying proof of the two main results concerning the validity of Chavel's conjecture on the domain monotonicity of the Neumann heat kernel, due to I. Chavel ([12]), respectively W. S. Kendall ([16]).

1. Introduction

The technique of coupling of reflecting Brownian motions is a useful tool used by several authors in connection to the study of the Neumann heat kernel of the corresponding domain (see [2], [3], [6], [11], [16], [17], etc).

In a series of paper, Krzysztof Burdzy et. al. ([1], [2], [3], [6], [10],) introduced the *mirror coupling* of reflecting Brownian motions in a smooth domain $D \subset \mathbb{R}^d$ and used it in order to derive properties of eigenvalues and eigenfunctions of the Neumann Laplaceian on D.

In the present paper, we show that the mirror coupling can be extended to the case when the two reflecting Brownian motions live in different domains $D_1, D_2 \subset \mathbb{R}^d$.

The main difficulty in the extending the construction of the mirror coupling comes from the fact that the stochastic differential equation(s) describing the mirror coupling has a singularity at the times when coupling occurs. In the case $D_1 = D_2 = D$ considered by Burdzy et. all. this problem is not a major problem (although the technical details are quite involved, see [2]), since after the coupling time the processes move together. In the case $D_1 \neq D_2$ however, this is a major problem: after processes have coupled, it is possible for them to decouple (for example in the case when the processes are coupled and they hit the boundary of one of the domains).

It is worth mentioning that the method used for proving the existence of the solution is new, and it relies on the additional hypothesis that the smaller domain

2000 *Mathematics Subject Classification.* Primary 60J65, 60H20. Secondary 35K05, 60H30.

Key words and phrases. couplings, mirror coupling, reflecting Brownian motion, Chavel's conjecture.

The author kindly acknowledges the support from CNCSIS-UEFISCSU research grant PNII - IDEI 209.

D_2 (or more generally $D_1 \cap D_2$) is a convex domain. This hypothesis allows us to construct an explicit set of solutions in a sequence of approximating polygonal domains for D_2, which converge to the desired solution.

As an application of the construction, we will derive a unifying proof of the two most important results on the challenging Chavel's conjecture on the domain monotonicity Neumann heat kernel ([12], [16]), which also gives a possible new line of approach for this conjecture (note that by the results in [4], Chavel's conjecture does not hold in its full generality, but the additional hypotheses under which this conjecture holds are not known at the present moment).

The structure of the paper is as follows: in Section 2 we briefly describe the construction of Burdzy et. al. of the mirror coupling in a smooth bounded domain domain $D \subset \mathbb{R}^d$.

In Section 3, in Theorem 3.1, we give the main result which shows that the mirror coupling can be extended to the case when $\overline{D_2} \subset D_1$ are smooth bounded domains in $\mathbb{R}^d$ and D_2 is convex (some extensions of the theorem are presented in Section 5).

Before proceeding with the proof of theorem, in Remark 3.4 we show that the proof of the theorem can be reduced to the case when $D_1 = \mathbb{R}^d$. Next, in Section 3.1, we show that in the case $D_2 = (0, \infty) \subset D_1 = R$ the solution is essentially given by Tanaka's formula (Remark 3.5), and then we give the proof of the main theorem in the 1-dimensional case (Proposition 3.6).

In Section 3.2, we first prove the existence of the mirror coupling in the case when D_2 is a half-space in $\mathbb{R}^d$ and $D_1 = \mathbb{R}^d$ (Lemma 3.8), and then we use this result in order to prove the existence of the mirror coupling in the case when D_2 is a convex polygonal domain in $\mathbb{R}^d$ and $D_1 = \mathbb{R}^d$ (Theorem 3.9). Some of the properties of coupling, essential for the extension to the general case are detailed in Proposition 3.10.

In Section 4 we give the proof of the main Theorem 3.1. The idea of the proof is to construct a sequence (Y_t^n, X_t) of mirror couplings in (D_n, D_1), where $D_n \nearrow D_2$ is a sequence of convex polygonal domains in $\mathbb{R}^d$, and to use the properties of the mirror couplings in polygonal domains (Proposition 3.10) in order to show that the sequence Y_t^n converges to a process Y_t, which gives the desired solution to the problem.

The last section of the paper (Section 5) is devoted to discussing the applications and the extensions of the mirror coupling constructed in Theorem 3.1. First, in Theorem 5.3 we use the mirror coupling in order to give a simple, unifying proof of the results of I. Chavel and W. S. Kendall on the domain monotonicity of the Neumann heat kernel (Chavel's Conjecture 5.1). The proof is probabilistic in spirit, relying on the geometric properties of the mirror coupling.

In Remark 5.5, we discuss the equivalent analytic counterpart of the proof Theorem 5.3, which might give a possible new line of approach for extending Chavel's conjecture to other classes of domains.

Without giving all the technical details, we discuss the extension of the mirror coupling to other classes of domains (smooth bounded domains $D_{1,2} \subset \mathbb{R}^d$ with non-tangential boundaries, such that $D_1 \cap D_2$ is a convex domain).

The paper concludes with a discussion on the (non) uniqueness of the mirror coupling. It is shown here that the lack of uniqueness is due to the fact that after

coupling, the processes might decouple, not only on the boundary of the domain, but even when they are inside of it.

The two basic solutions give rise to the *sticky*, respectively *non-sticky* mirror couplings, and there is a whole range of intermediate possibilities. The stickiness refers to the fact that after coupling the processes "stick" to each other as long as possible (this is the coupling constructed in Theorem 3.1) hence the name "sticky" mirror coupling, or they can split apart immediately after coupling, in the case of "non-sticky" mirror coupling, the general case (*weak/mild* mirror couplings) being a mixture of these two basic behaviors.

We developed the extension of the mirror coupling having in mind the application to Chavel's conjecture, for which the sticky mirror coupling is the "right" tool, but perhaps the other mirror couplings (the non-sticky and the mild mirror couplings) might prove useful in other applications.

2. Mirror couplings of reflecting Brownian motions

Reflecting Brownian motion in a smooth domain $D \subset \mathbb{R}^d$ can be defined as a solution of the stochastic differential equation

$$X_t = x + B_t + \int_0^t \nu_D(X_s)\, dL_s^X, \tag{2.1}$$

where B_t is a d-dimensional Brownian motion, ν_D is the inward unit normal vector field on ∂D and L_t^X is the boundary local time of X_t (the continuous non-decreasing process which increases only when $X_t \in \partial D$).

In [1], the authors introduced the *mirror coupling* of reflecting Brownian motion in a smooth domain $D \subset \mathbb{R}^d$ (piecewise C^2 domain in $\mathbb{R}^2$ with a finite number of convex corners or a C^2 domain in $\mathbb{R}^d$, $d \geq 3$).

They considered the following system of stochastic differential equations:

$$X_t = x + W_t + \int_0^t \nu_D(X_s)\, dL_s^X \tag{2.2}$$

$$Y_t = y + Z_t + \int_0^t \nu_D(X_s)\, dL_s^Y \tag{2.3}$$

$$Z_t = W_t - 2\int_0^t \frac{Y_s - X_s}{\left\|Y_s - X_s\right\|^2}(Y_s - X_s)\cdot dW_s \tag{2.4}$$

for $t < \xi$, where $\xi = \inf\{s > 0 : X_s = Y_s\}$ is the coupling time of the processes, after which the processes X and Y evolve together, i.e. $X_t = Y_t$ for $t \geq \xi$ and $Z_t = Z_\xi + 1_{t\geq\xi}(W_t - W_\xi)$.

In the notation of [1], considering the Skorokhod map $\Gamma : C\left([0,\infty) : \mathbb{R}^d\right) \to C\left([0,\infty) : \bar{D}\right)$, we have $X = \Gamma(x + W)$ and $Y = \Gamma(y + Z)$, and the above system reduces to

$$Z_t = \int_0^{t\wedge\xi} G\left(\Gamma(y+Z)_s - \Gamma(x+W)_s\right) dW_s + 1_{t\geq\xi}(W_t - W_\xi), \tag{2.5}$$

where $\xi = \inf\{t \geq 0 : \Gamma(x + W_s) = \Gamma(y + Z_s)\}$, for which the authors proved the pathwise uniqueness and the strong uniqueness of the process Z_t (given the Brownian motion W_t).

In the above $G:\mathbb{R}^d\to\mathcal{M}_{d\times d}$ denotes the function defined by

$$G(z)=\begin{cases} H\left(\frac{z}{|z|}\right), & \text{if } z\neq 0\\ 0, & \text{if } z=0\end{cases} \tag{2.6}$$

where for a unitary vector $m\in\mathbb{R}^d$, $H(m)$ represents the linear transformation given by the $d\times d$ matrix

$$H(m)=I-2m\,m', \tag{2.7}$$

that is

$$H(m)\,v=v-2\,(m\cdot v)\,m \tag{2.8}$$

is the mirror image of $v\in\mathbb{R}^d$ with respect to the hyperplane through the origin perpendicular to m (m' denotes the transpose of the vector m, vectors being considered as column vectors).

The pair $(X_t,Y_t)_{t\geq 0}$ constructed above is called a *mirror coupling* of reflecting Brownian motions in D starting at $x,y\in\bar{D}$.

Remark 2.1. The relation (2.4) can be written

$$dZ_t=G\left(\frac{X_t-Y_t}{||X_t-Y_t||}\right)dW_t,$$

which shows that for $t<\xi$ the increments of Z_t are mirror images of the increments of W_t with respect to the line of symmetry M_t of X_t and Y_t, which justifies the name of *mirror coupling*.

3. Extension of the mirror coupling

The main contribution of the author is the observation that the mirror coupling introduced above can be extended to the case when the two reflecting Brownian motion have different state spaces, that is when X_t is a reflecting Brownian motion in D_1 and Y_t is a reflecting Brownian motion in D_2. Although the construction can be carried out in a more general setup (see the concluding remarks in Section 5), in the present section we will restrict to the case when one of the domains is strictly contained in the other one.

The main result is the following:

Theorem 3.1. *Let $D_{1,2}\subset\mathbb{R}^d$ be smooth bounded domains (piecewise C^2-smooth boundary with convex corners in $\mathbb{R}^2$, or C^2-smooth boundary in $\mathbb{R}^d$, $d\geq 3$ will suffice) with $\overline{D_2}\subset D_1$ and D_2 convex domain, and let $x\in\bar{D}_1$ and $y\in\bar{D}_2$ be arbitrarily fixed points. Given a d-dimensional Brownian motion $(W_t)_{t\geq 0}$ starting at 0 on a probability space $(\Omega,\mathcal{F},P)$, there exists a strong solution of the following system of stochastic differential equations*

$$X_t = x+W_t+\int_0^t \nu_{D_1}(X_s)\,dL_s^X \tag{3.1}$$

$$Y_t = y+Z_t+\int_0^t \nu_{D_2}(Y_s)\,dL_s^Y \tag{3.2}$$

$$Z_t = \int_0^t G(Y_s-X_s)\,dW_s \tag{3.3}$$

or equivalent

$$Z_t = \int_0^t G\left(\tilde{\Gamma}\left(y+Z\right)_s - \Gamma\left(x+W\right)_s\right) dW_s, \tag{3.4}$$

where Γ *and* $\tilde{\Gamma}$ *denote the corresponding Skorokhod maps which define the reflecting Brownian motion* $X = \Gamma\left(x+W\right)$ *in* D_1*, respectively* $Y = \tilde{\Gamma}\left(y+Z\right)$ *in* D_2*, and* $G : \mathbb{R}^d \to \mathcal{M}_{d\times d}$ *denotes the following modification of the function* G *defined in the previous section:*

$$G\left(z\right) = \begin{cases} H\left(\frac{z}{\|z\|}\right), & \text{if } z \neq 0 \\ I, & \text{if } z = 0 \end{cases}. \tag{3.5}$$

Remark 3.2. As it will follow from the proof of the theorem, with the choice of G above, in the case $D_1 = D_2 = D$ the solution of the equation (3.4) given by the theorem above is the same as the solution of the equation (2.5) considered by the authors in [1] (as pointed out by the authors, the choice of $G\left(0\right)$ is irrelevant in this case), and therefore the above theorem is a natural generalization of their result to the case when the two processes live in different spaces. We will refer to a solution X_t, Y_t given by the above theorem as a *mirror coupling* of reflecting Brownian motions in D_1, respectively D_2, starting from $(x, y) \in \overline{D_1} \times \overline{D_2}$ with driving Brownian motion W_t.

As we will see in Section 5, without additional assumptions, the solution of (3.4) is not pathwise unique. This is due to the fact that the stochastic differential equation has a singularity at the origin (i.e. at times when the coupling occurs); the general mirror coupling can be thought as depending on a parameter which is a measure of the stickiness of the coupling: once the processes X_t and Y_t have coupled, they can either move together until one of them hits the boundary (*sticky* mirror coupling - this is in fact the solution constructed in the above theorem), or they can immediately split after coupling (non-sticky mirror coupling), and there is a whole range of intermediate possibilities (see the discussion at the end of Section 5).

As an application, in Section 5 we will use the former mirror coupling (the sticky mirror coupling) to give a unifying proof of Chavel's conjecture on the domain monotonicity of the Neumann heat kernel for domains $D_{1,2}$ satisfying the ball condition, although the other possible choices for the mirror coupling might prove useful in other contexts.

Before carrying out the proof, we begin with some preliminary remarks which will allow us to reduce the proof of the above theorem to the case $D_1 = \mathbb{R}^d$.

Remark 3.3. The main difference from the case when $D_1 = D_2 = D$ considered by the authors in [1] is that after the coupling time ξ the processes X_t and Y_t may decouple. For example, if $t \geq \xi$ is a time when $X_t = Y_t \in \partial D_2$, the process Y_t being conditioned to stay in $\overline{D_2}$, receives a push in the direction of the inward unit normal to the boundary of D_2, while the process X_t behaves like a free Brownian motion near this point (we assumed that D_2 is strictly contained in D_1), and therefore the processes X and Y will drift apart, that is they will *decouple*. Also, as shown in Section 5, because the function G has a discontinuity at the origin, it is possible that the solutions decouple even inside the domain D_2, so, without additional assumptions, the mirror coupling is not uniquely determined (there is no pathwise uniqueness of (3.4)).

Remark 3.4. To fix ideas, for an arbitrarily fixed $\varepsilon > 0$ chosen small enough such that $\varepsilon < \operatorname{dist}(\partial D_1, \partial D_2)$, we consider the sequence $(\xi_n)_{n\geq 1}$ of *coupling times* and the sequence $(\tau_n)_{n\geq 0}$ of times when the processes are ε-decoupled (ε-*decoupling times*, or simply *decoupling times* by an abuse of language) defined inductively by

$$\begin{aligned} \xi_n &= \inf\{t > \tau_{n-1} : X_t = Y_t\}, \\ \tau_n &= \inf\{t > \xi_n : |X_t - Y_t| > \varepsilon\}, \end{aligned}$$

where $\tau_0 = 0$ and $\xi_1 = \xi$ is the first coupling time.

To construct the general mirror coupling (that is, to prove the existence of a solution to (3.1)-(3.3) above, or equivalent to (3.4)), we proceed as follows.

First note that on the time interval $[0, \xi]$, the arguments used in the proof of Theorem 2 in [1] (pathwise uniqueness and the existence of a strong solution Z of (3.4)) do not rely on the fact that $D_1 = D_2$, hence the same arguments can be used to prove the existence of a strong solution of (3.4) on the time interval $[0, \xi_1] = [0, \xi]$. Indeed, given W_t, (3.1) has a strong solution which is pathwise unique (the reflecting Brownian motion X_t in D_1), and therefore the proof of pathwise uniqueness and the existence of a strong solution of (3.4) is the same as in [1] considering $D = D_2$. Also note that as pointed by the authors, the value $G(0)$ is irrelevant in their proof, since the problem is constructing the processes until they meet, that is for $Y_t - X_t \neq 0$, for which the definition of G coincides with (3.5).

Next, assuming the existence of a strong solution to (3.4) on $[\xi_1, \tau_1]$ (and therefore on $[0, \tau_1]$), since at time τ_1 the processes are $\varepsilon > 0$ units apart, we can apply again the results in [1] (with $\tilde{B}_t = B_{t+\tau_1} - B_{\tau_1}$ and starting points X_{τ_1} and Y_{τ_1}) in order to obtain a strong solution of (3.4) on the time interval $[\tau_1, \xi_2]$, and therefore by patching we obtain the existence of a strong solution of (3.4) on the time interval $[0, \xi_2]$.

For an arbitrarily fixed $t > 0$, since only a finite number of coupling/decoupling times ξ_n and τ_n can occur in the time interval $[0, t]$ (we use here the fact that D_2 is strictly contained in D_1), it follows that there exists a strong solution to (3.4) on $[0, t]$ for any $t > 0$ (and therefore on $[0, \infty)$), provided we show the existence of a strong solution of (3.4) on $[\xi_n, \tau_n]$, $n \geq 1$.

In order to prove this claim, it suffices therefore to show that for any starting points $x = y \in \bar{D}_2$ of the mirror coupling, there exists a strong solution to (3.4) until the ε-decoupling time τ_1. Since $\varepsilon < \operatorname{dist}(\partial D_1, \partial D_2)$, it follows that the process X_t cannot reach the boundary ∂D_1 before the ε-decoupling time τ_1, and therefore we can consider that X_t is a free Brownian motion in $\mathbb{R}^d$, that is we can reduce the proof of Theorem 3.1 to the case when $D_1 = \mathbb{R}^d$.

We will first give the proof of the in the 1-dimensional case, then we will extend the construction to polygonal domains in $\mathbb{R}^d$, and we will conclude with the proof in the general case.

3.1. **The 1-dimensional case.** From Remark 3.4 it follows that in order to construct the mirror coupling in the 1-dimensional case, it suffices to consider $D_1 = \mathbb{R}$ and $D_2 = (0, a)$ and to construct a strong solution for $t \leq \tau_1 = \inf\{s > 0 : |X_s - Y_s| > \varepsilon\}$

of the following system:

$$X_t = x + W_t \tag{3.6}$$

$$Y_t = x + Z_t + L_t^Y \tag{3.7}$$

$$Z_t = \int_0^t G(Y_s - X_s)\,dW_s \tag{3.8}$$

for an arbitrary choice $x \in [0, a]$ of the starting point of the mirror coupling, where $\varepsilon \in (0, a)$ is arbitrarily small, $(W_t)_{t\geq 0}$ is a 1-dimensional Brownian motion starting at $W_0 = 0$ and the function $G : \mathbb{R} \to \mathcal{M}_{1\times 1} \equiv \mathbb{R}$ is given in this case by

$$G(x) = \begin{cases} -1, & \text{if } x \neq 0 \\ +1, & \text{if } x = 0 \end{cases}.$$

Remark 3.5. Before proceeding with the proof, it is worth mentioning that the heart of the construction is Tanaka's formula. To see this, consider for the moment $a = \infty$, and note that Tanaka formula

$$|x + W_t| = x + \int_0^t \operatorname{sgn}(x + W_s)\,dW_s + L_t^0(x + W)$$

gives a representation of the reflecting Brownian motion $|x + W_t|$ in which the increments of the martingale part of $|x + W_t|$ are the increments of W_t when $x + W_t \in [0, \infty)$, respectively the opposite (minus) of the increments of W_t in the opposite case ($L_t^0(x + W)$ denotes here the local time at 0 of $x + W_t$).

Noting that the condition $x + W_t \in [0, \infty)$ is the same as $|x + W_t| = x + W_t$, from the definition of the function G it follows that the above can be written in the form

$$|x + W_t| = x + \int_0^t G(|x + W_s| - (x + W_s))\,dW_s + L_t^{x+W},$$

which shows that a strong solution to (3.6) - (3.8) above (in the case $a = \infty$) is given explicitly by $X_t = x + W_t$ and $Y_t = |x + W_t|$ (and $Z_t = \int_0^t \operatorname{sgn}(x + W_s)\,dW_s$).

We have the following:

Proposition 3.6. *Given a* 1*-dimensional Brownian motion* $(W_t)_{t\geq 0}$ *starting at* $W_0 = 0$*, a strong solution to (3.6) – (3.8) above for* $t < \tau_1 = \inf\{s > 0 : |X_s - Y_s| > \varepsilon\}$ *is given by*

$$\begin{cases} X_t = x + W_t \\ Y_t = |a - |x + W_t - a|| \\ Z_t = \int_0^t \operatorname{sgn}(W_s)\operatorname{sgn}(a - W_s)\,dW_s \end{cases},$$

where

$$\operatorname{sgn}(x) = \begin{cases} +1, & \textit{if } x \geq 0 \\ -1, & \textit{if } x < 0 \end{cases}.$$

Proof. Since $\varepsilon < a$, it follows that for $t \leq \tau_1$ we have $X_t = x + W_t \in (-a, 2a)$, and therefore

$$Y_t = |a - |x + W_t - a|| = \begin{cases} -(x + W_t), & x + W_t \in (-a, 0) \\ x + W_t, & x + W_t \in [0, a] \\ 2a - x - W_t, & x + W_t \in (a, 2a) \end{cases}. \tag{3.9}$$

Applying the Tanaka-Itô formula to the function $f(z) = |a - |z-a||$ and to the Brownian motion $X_t = x + W_t$, for $t \le \tau_1$ we obtain

$$\begin{aligned} Y_t &= x + \int_0^t \operatorname{sgn}(x+W_s)\operatorname{sgn}(a-x-W_s)\, d(x+W_s) + L_t^0 - L_t^a \\ &= x + \int_0^t \operatorname{sgn}(x+W_s)\operatorname{sgn}(a-x-W_s)\, dW_s + \int_0^t \nu_{D_2}(Y_s)\, d\left(L_s^0 + L_s^a\right), \end{aligned}$$

where $L_t^0 = \sup_{s\le t}(x+W_s)^-$ and $L_t^a = \sup_{s\le t}(x+W_s-a)^+$ are the local times of $x+W_t$ at 0, respectively at a, and $\nu_{D_2}(0) = +1$, $\nu_{D_2}(a) = -1$.

From (3.9) and the definition of G we have

$$\begin{aligned} \operatorname{sgn}(x+W_s)\operatorname{sgn}(a-x-W_s) &= \begin{cases} -1, & x+W_s \in (-a,0) \\ +1, & x+W_s \in [0,a] \\ -1, & x+W_s \in (a,2a) \end{cases} \\ &= \begin{cases} +1, & X_s = Y_s \\ -1, & X_s \ne Y_s \end{cases} \\ &= G(Y_s - X_s), \end{aligned}$$

and therefore the previous formula can be written equivalently

$$Y_t = x + Z_t + \int_0^t \nu_{D_2}(Y_s)\, dL_s^Y,$$

where

$$Z_t = \int_0^t G(Y_s - X_s)\, dW_s$$

and $L_t^Y = L_t^0 + L_t^a$ is a continuous nondecreasing process which increases only when $x + W_t \in \{0, a\}$, that is only when $Y_t \in \partial D_2$, which concludes the proof. □

3.2. **The case of polygonal domains.** In this section we will consider the case when $D_2 \subset D_1 \subset \mathbb{R}^d$ are convex polygonal domains (convex domains bounded by hyperplanes in $\mathbb{R}^d$). From Remark 3.4 it follows that we can consider $D_1 = \mathbb{R}^d$ and therefore it suffices to prove the existence of a strong solution to

$$X_t = X_0 + W_t \tag{3.10}$$

$$Y_t = Y_0 + Z_t + \int_0^t \nu_{D_2}(Y_s)\, dL_s^Y \tag{3.11}$$

$$Z_t = \int_0^t G(Y_s - X_s)\, dW_s \tag{3.12}$$

or equivalently

$$Z_t = \int_0^t G\left(\tilde{\Gamma}(Y_0 + Z)_s - X_0 - W_s\right) dW_s, \tag{3.13}$$

where W_t is a d-dimensional Brownian motion starting at $W_0 = 0$ and $X_0 = Y_0 \in \bar{D}_2$.

The construction relies on the skew product representation of Brownian motion in spherical coordinates, that is

$$X_t = R_t \Theta_{\sigma_t}, \tag{3.14}$$

where $R_t = |X_t| \in \mathrm{BES}\,(d)$ is a Bessel process of order d and $\Theta_t \in \mathrm{BM}\left(S^{d-1}\right)$ is an independent Brownian motion on the unit sphere S^{d-1} in $\mathbb{R}^d$, run at speed

$$\sigma_t = \int_0^t \frac{1}{R_s^2} ds, \tag{3.15}$$

which depends only on R_t.

Remark 3.7. One way to construct the Brownian motion $\Theta_t = \Theta_t^{d-1}$ on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is to proceed inductively on $d \geq 2$, using the skew product representation of Brownian motion on the sphere $\Theta_t^{d-1} \in S^{d-1}$ (see [15])

$$\Theta_t^{d-1} = \left(\cos\theta_t^1, \sin\theta_t^1 \Theta_{\alpha_t}^{d-2}\right)$$

where $\theta^1 \in \mathrm{LEG}\,(d-1)$ is a Legendre process of order $d-1$ on $[0,\pi]$, and $\Theta_t^{d-2} \in S^{d-2}$ is an independent Brownian motion on S^{d-2}, run at speed

$$\alpha_t = \int_0^t \frac{1}{\sin^2\theta_s^1} ds.$$

Therefore, considering independent processes $\theta_t^1, \dots \theta_t^{d-1}$, where $\theta^i \in \mathrm{LEG}\,(d-i)$ on $[0,\pi]$ $(i = \overline{1, d-2})$ and θ_t^{d-1} a 1-dimensional Brownian ($\Theta_t^1 = \left(\cos\theta_t^1, \sin\theta_t^1\right) \in S^1$ is a Brownian motion on S^1), we have

$$\Theta_t^{d-1} = \left(\cos\theta_t^1, \sin\theta_t^1\cos\theta_t^2, \sin\theta_t^1\sin\theta_t^2\cos\theta_t^3, \dots, \sin\theta_t^1\cdots\sin\theta_t^{d-1}\sin\theta_t^{d-1}\right),$$

or equivalent, in spherical coordinates, $\Theta_t^{d-1} \in S^{d-1}$ is given by

$$\Theta_t^{d-1} = \left(\theta_t^1, \dots, \theta_t^{d-2}, \theta_t^{d-1}\right). \tag{3.16}$$

To construct the solution we first consider first the case when D_2 is a half-space $\mathcal{H}_d^+ = \left\{\left(z^1, \dots, z^d\right) \in \mathbb{R}^d : z^d > 0\right\}$.

Given an angle $\varphi \in \mathbb{R}$, we introduce the rotation matrix $R(\varphi) \in \mathcal{M}_{d\times d}$ which leaves invariant the first $d-2$ coordinates and rotates clockwise by the angle α the remaining 2 coordinates, that is

$$R(\alpha) = \begin{pmatrix} 1 & & 0 & 0 & 0 \\ & \ddots & & \dots & \dots \\ 0 & & 1 & 0 & 0 \\ 0 & \cdots & 0 & \cos\varphi & -\sin\varphi \\ 0 & \cdots & 0 & \sin\varphi & \cos\varphi \end{pmatrix} \tag{3.17}$$

We have the following:

Lemma 3.8. *Let $D_2 = \mathcal{H}_d^+ = \left\{\left(z^1, \dots, z^d\right) \in \mathbb{R}^d : z^d > 0\right\}$ and assume that*

$$Y_0 = R(\varphi_0) X_0 \tag{3.18}$$

for some $\varphi_0 \in \mathbb{R}$.

Consider the reflecting Brownian motion $\tilde{\theta}_t^{d-1}$ on $[0,\pi]$ with driving Brownian motion θ_t^{d-1}, where θ_t^{d-1} is the $(d-1)$ spherical coordinate of $G(Y_0 - X_0) X_t$, given by (3.14) – (3.16) above, that is:

$$\tilde{\theta}_t^{d-1} = \theta_t^{d-1} + L_t^0\left(\tilde{\theta}^{d-1}\right) - L_t^\pi\left(\tilde{\theta}^{d-1}\right), \qquad t \geq 0,$$

and $L_t^0\left(\tilde{\theta}^{d-1}\right)$, $L_t^\pi\left(\tilde{\theta}^{d-1}\right)$ represent the local times of $\tilde{\theta}^{d-1}$ at 0, respectively at π.

A strong solution to (3.10) - (3.12) above is explicitly given by

$$Y_t = \begin{cases} R(\varphi_t)\, G(Y_0 - X_0)\, X_t, & t < \xi \\ |X_t|_d\,, & t \geq \xi \end{cases} \tag{3.19}$$

where $\xi = \inf\{t > 0 : X_t = Y_t\}$ *is the coupling time, the rotation angle* φ_t *is given by*

$$\varphi_t = L_t^0\left(\tilde{\theta}^{d-1}\right) - L_t^{\pi}\left(\tilde{\theta}^{d-1}\right), \qquad t \geq 0,$$

and for $z = \left(z^1, z^2 \ldots, z^d\right) \in \mathbb{R}^d$ *we denoted by* $|z|_d = \left(z^1, z^2, \ldots, \left|z^d\right|\right)$.

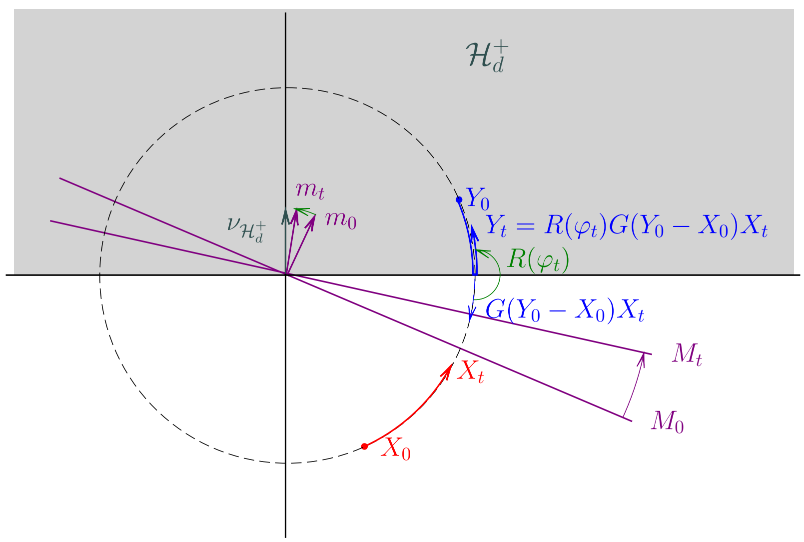

FIGURE 1. The mirror coupling of a free Brownian motion X_t and a reflecting Brownian motion Y_t in the half-space $\mathcal{H}_d^+$.

Proof. Recall that for $m \in \mathbb{R}^d - \{0\}$, $G(m)\, v$ denotes the mirror image (symmetric) of $v \in \mathbb{R}^d$ in the hyperplane through the origin, perpendicular to m.

By Itô formula, we have

$$Y_{t\wedge\xi} = Y_0 + \int_0^{t\wedge\xi} R(\varphi_s)\, G(Y_0 - X_0)\, dX_s + \int_0^{t\wedge\xi} R\left(\varphi_s + \frac{\pi}{2}\right) G(Y_0 - X_0)\, dL_s \tag{3.20}$$

Note that the composition $R \circ G$ (a symmetry followed by a rotation) is a symmetry, and since $|Y_t| = |X_t|$ for all $t \geq 0$, it follows that X_t and Y_t are symmetric with respect to a hyperplane passing through the origin for all $t \leq \xi$; therefore, from the definition (3.5) of G it follows that we have $Y_t = G(Y_t - X_t)\, X_t$ for all $t \leq \xi$.

Also note that when $L_s^0\left(\tilde{\theta}^{d-1}\right)$ increases, $Y_s \in \partial D_2$ and we have

$$R\left(\varphi_s + \frac{\pi}{2}\right) G(Y_0 - X_0)\, X_s = R\left(\frac{\pi}{2}\right) Y_s = \nu_{D_2}(Y_s)$$

and if $L_s^\pi\left(\tilde{\theta}^{d-1}\right)$ increases, $Y_s \in \partial D_2$ and we have

$$R\left(\varphi_s + \frac{\pi}{2}\right) G\left(Y_0 - X_0\right) X_s = R\left(\frac{\pi}{2}\right) Y_s = -\nu_{D_2}\left(Y_s\right).$$

It follows that the relation (3.20) above can be written in the form

$$Y_{t\wedge\xi} = Y_0 + \int_0^{t\wedge\xi} G\left(Y_s - X_s\right) dX_s + \int_0^{t\wedge\xi} \nu_{D_2}\left(Y_s\right) dL_s^Y,$$

where $L_t^Y = L_t^0\left(\tilde{\theta}^{d-1}\right) + L_t^\pi\left(\tilde{\theta}^{d-1}\right)$ is a continuous non-decreasing process which increases only when $Y_t \in \partial D_2$, and therefore Y_t given by (3.19) is a strong solution to (3.10) - (3.12) for $t \leq \xi$.

For $t \geq \xi$, we have $Y_t = |X_t|_d = \left(X_t^1, X_t^2, \ldots, \left|X_t^d\right|\right)$, and similar to the 1-dimensional case, by Tanaka formula we obtain:

$$\begin{aligned} Y_{t\vee\xi} &= Y_\xi + \int_\xi^{t\vee\xi} \left(1,\ldots,1,\operatorname{sgn}\left(X_s^d\right)\right) dX_s + \int_\xi^{t\vee\xi} (0,0,\ldots,1)\, L_t^0\left(X^d\right) \\ &= Y_\xi + \int_\xi^{t\vee\xi} G\left(Y_s - X_s\right) dX_s + \int_\xi^{t\vee\xi} \nu_{D_2}\left(Y_s\right) L_t^Y, \end{aligned} \tag{3.21}$$

since

$$\begin{aligned} G\left(Y_s - X_s\right) &= \begin{cases} (1,\ldots,1,+1), & X_s = Y_s \\ (1,\ldots,1,-1), & X_s \neq Y_s \end{cases} \\ &= \begin{cases} (1,\ldots,1,+1), & X_s^d \geq 0 \\ (1,\ldots,1,-1), & X_s^d < 0 \end{cases} \\ &= \left(1,\ldots,1,\operatorname{sgn}\left(X_s^d\right)\right). \end{aligned}$$

$L_t^Y = L_t^0\left(X^d\right)$ in (3.21) is a continuous non-decreasing process which increases only when $Y_t \in \partial D_2$ (we denoted by $L_t^0\left(X^d\right)$ the local time at 0 of the last cartesian coordinate X^d of X), which shows that Y_t also solves (3.10) - (3.12) for $t \geq \xi$, and therefore Y_t is a strong solution to (3.10) - (3.12) for $t \geq 0$, concluding the proof. □

Consider now the case of a general polygonal domain $D_2 \subset \mathbb{R}^d$. We will show that a strong solution to (3.10) - (3.12) can be constructed from the previous lemma, by choosing the appropriate coordinate system.

Consider the times $(\sigma_n)_{n\geq 0}$ at which the solution Y_t hits different bounding hyperplanes of ∂D_2, that is $\sigma_0 = \inf\{s \geq 0 : Y_s \in \partial D_2\}$ and inductively

$$\sigma_{n+1} = \inf\left\{t \geq \sigma_n : \begin{array}{c} Y_t \in \partial\mathcal{D}_2 \text{ and } Y_t, Y_{\sigma_n} \text{ belong to different}^1 \\ \text{bounding hyperplanes of } \partial D_2 \end{array}\right\}. \tag{3.22}$$

If $X_0 = Y_0 \in \partial D_2$ belong to a certain bounding hyperplane of D_2, we can chose the coordinate system so that this hyperplane is $\mathcal{H}_d = \left\{\left(z^1,\ldots,z^d\right) \in \mathbb{R}^d : z^d = 0\right\}$ and $D_2 \subset \mathcal{H}_d^+$, and we let $\mathcal{H}_d$ be any bounding hyperplane of D_2 otherwise.

Then, on the time interval $[\sigma_0, \sigma_1)$, the strong solution to (3.10) - (3.12) is given explicitly by (3.19) in Lemma 3.8.

If $\sigma_1 < \infty$, we distinguish two cases: $X_{\sigma_1} = Y_{\sigma_1}$ and $X_{\sigma_1} \neq Y_{\sigma_1}$. Let $\mathcal{H}$ denote the bounding hyperplane of D which contains Y_{σ_1}, and let $\nu_{\mathcal{H}}$ denote the unit normal to $\mathcal{H}$ pointing inside D_2.

If $X_{\sigma_1} = Y_{\sigma_1} \in \mathcal{H}$, choosing again the coordinate system conveniently, we may assume that $\mathcal{H}$ is the hyperplane is $\mathcal{H}_d = \left\{\left(z^1,\ldots,z^d\right) \in \mathbb{R}^d : z^d = 0\right\}$, and on

the time interval $[\sigma_1, \sigma_2)$ the coupling $(X_{\sigma_1+t}, Y_{\sigma_1+t})_{t\in[0,\sigma_2-\sigma_1)}$ is given again by Lemma 3.8.

If $X_{\sigma_1} \neq Y_{\sigma_1} \in \mathcal{H}$, in order to apply the lemma, we have to show that we can choose the coordinate system so that the condition (3.18) holds. If $Y_{\sigma_1} - X_{\sigma_1}$ is a vector perpendicular to $\mathcal{H}$, by choosing the coordinate system so that $\mathcal{H} = \mathcal{H}_d = \left\{\left(z^1, \ldots, z^d\right) \in \mathbb{R}^d : z^d = 0\right\}$, the problem reduces to the 1-dimensional case (the first $d-1$ coordinates of X and Y are the same), and it can be handled as in Proposition 3.6 by the Tanaka formula. The proof being similar, we omit it.

If $X_{\sigma_1} \neq Y_{\sigma_1} \in \mathcal{H}$ and $Y_{\sigma_1} - X_{\sigma_1}$ is not orthogonal to $\mathcal{H}$, consider $\tilde{X}_{\sigma_1} = \mathrm{pr}_{\mathcal{H}} X_{\sigma_1}$ the projection of X_{σ_1} onto $\mathcal{H}$, and therefore $\tilde{X}_{\sigma_1} \neq Y_{\sigma_1}$. The plane of symmetry of X_{σ_1} and Y_{σ_1} intersects the line determined by $\tilde{X}_{\sigma_1}$ and Y_{σ_1} at a point, and we consider this point as the origin of the coordinate system (note that the intersection cannot be empty, for otherwise the vectors $Y_{\sigma_1} - X_{\sigma_1}$ and $Y_{\sigma_1} - \tilde{X}_{\sigma_1}$ were parallel, which is impossible since then $Y_{\sigma_1} - X_{\sigma_1}, Y_{\sigma_1} - \tilde{X}_{\sigma_1}$ and $Y_{\sigma_1} - \tilde{X}_{\sigma_1}, X_{\sigma_1} - \tilde{X}_{\sigma_1}$ were perpendicular pairs of vectors, contradicting $\tilde{X}_{\sigma_1} \neq Y_{\sigma_1}$ - see Figure 2).

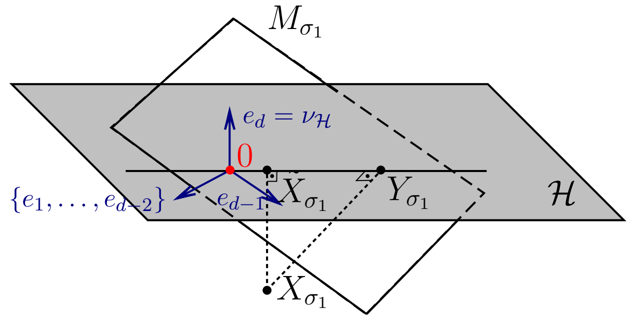

FIGURE 2. Construction of an appropriate coordinate system.

Choose a orthonormal basis $\{e_1, \ldots, e_d\}$ in $\mathbb{R}^d$ such that $e_d = \nu_{\mathcal{H}}$ is the normal vector to $\mathcal{H}$ pointing inside D_2, $e_{d-1} = \frac{1}{\left|Y_{\sigma_1} - X_{\sigma_1}\right|}\left(Y_{\sigma_1} - X_{\sigma_1}\right)$ is a unit vector lying in the 2-dimensional plane determined by the origin and the vectors e_d and $Y_{\sigma_1} - X_{\sigma_1}$, and $\{e_1, \ldots, e_{d-2}\}$ is a completion of $\{e_{d-1}, e_d\}$ to a orthonormal basis in $\mathbb{R}^d$ (see Figure 2).

Note that by the construction, the vectors $e_1, \ldots, e_{d-2}$ are orthogonal to the 2-dimensional hyperplane containing the origin and the points X_{σ_1} and Y_{σ_1}, and therefore X_{σ_1} and Y_{σ_1} have the same (zero) first $d-2$ coordinates; also, since X_{σ_1} and Y_{σ_1} are at the same distance from the origin, it follows that Y_{σ_1} can be obtained from X_{σ_1} by a rotation which leaves invariant the first $d-2$ coordinates, which shows that the condition (3.18) of Lemma 3.8 is satisfied.

Since by construction the bounding hyperplane $\mathcal{H}$ of D_2 at Y_{σ_1} is given by $\mathcal{H}_d = \left\{\left(z^1, \ldots, z^d\right) \in \mathbb{R}^d : z^d = 0\right\}$ and $D_2 \subset \mathcal{H}_d^+ = \left\{\left(z^1, \ldots, z^d\right) \in \mathbb{R}^d : z^d > 0\right\}$, we can apply Lemma 3.8 and deduce that on the time interval $[\sigma_1, \sigma_2)$ a solution to (3.10) - (3.12) is given by $(X_{\sigma_1+t}, Y_{\sigma_1+t})_{t\in[0,\sigma_2-\sigma_1)}$.

Repeating the above argument we can construct inductively (in appropriate coordinate systems) the solution to (3.10) - (3.12) on any time interval $[\sigma_n, \sigma_{n+1})$, $n \geq 1$, therefore obtaining a strong solution to (3.10) - (3.12) defined for $t \geq 0$.

We summarize the above discussion in the following:

Theorem 3.9. *If $D_2 \subset \mathbb{R}^d$ is a convex polygonal domain, for any $X_0 = Y_0 \in \bar{D}_2$, there exists a strong solution to (3.10) - (3.12) above.*

Moreover, between successive hits of different bounding hyperplanes of D_2 (i.e. on each time interval $[\sigma_n, \sigma_{n+1})$ in the notation above) and for an appropriately chosen coordinate system, the solution is given by Lemma 3.8.

We will refer to the solution $(X_t, Y_t)_{t\geq 0}$ constructed in the previous theorem as a *mirror coupling* of reflecting Brownian motions in $\left(\mathbb{R}^d, D_2\right)$ with starting point $X_0 = Y_0 \in \overline{D_2}$.

If $X_t \neq Y_t$, the hyperplane M_t of symmetry between X_t and Y_t, passing through $\frac{X_t+Y_t}{2}$ with normal $m_t = \frac{1}{|Y_t - X_t|}(Y_t - X_t)$, will be referred to as the *mirror of the coupling*. For definiteness, when $X_t = Y_t$ we let M_t denote any hyperplane passing through $X_t = Y_t$, for example we choose M_t such that it is a left continuous function with respect to t.

Some of the properties of the mirror coupling are contained in the following:

Proposition 3.10. *If $D_2 \subset \mathbb{R}^d$ is a convex polygonal domain, for any $X_0 = Y_0 \in \bar{D}_2$, the mirror coupling given by the previous theorem has the following properties:*

i) *If the reflection takes place in the boundary hyperplane $\mathcal{H}$ of D_2 with inward unitary normal $\nu_{\mathcal{H}}$, then the angle $\angle(m_t;\nu_{\mathcal{H}})$ decreases monotonically to zero.*
ii) *When processes are not coupled, the mirror M_t lies outside D_2.*
iii) *Coupling can take place precisely when $X_t \in \partial D_2$. Moreover, if $X_t \in D_2$, then $X_t = Y_t$.*
iv) *If $D_\alpha \subset D_\beta$ are two polygonal domains and $(Y_t^\alpha; X_t)$, $(Y_t^\beta; X_t)$ are the corresponding mirror coupling starting from $x \in \overline{D_\alpha}$, for any $t > 0$ we have*

$$\sup_{s\leq t}\left|Y_s^\alpha - Y_s^\beta\right| \leq \operatorname{Dist}\left(D^\alpha, D^\beta\right) := \max_{\substack{x_\alpha\in\partial D_\alpha, x_\beta\in\partial D_\beta \\ (x_\beta - x_\alpha)\cdot\nu_{D_\alpha}(x_\alpha)\leq 0}} |x_\alpha - x_\beta| . \tag{3.23}$$

Proof. i) In the notation of Theorem 3.9, on the time interval $[\sigma_0, \sigma_1)$ we have $Y_t = X_t$ so $\angle(m_t, \nu_{\mathcal{H}}) = 0$, and the claim is verified in this case.

On an arbitrary time interval $[\sigma_n, \sigma_{n+1})$, in an appropriately chosen coordinate system, Y_t is given by Lemma 3.8. For $t < \xi$, Y_t is given by the rotation $R(\varphi_t)$ of $G(Y_0 - X_0)X_t$ which leaves invariant the first $(d-2)$ coordinates, and therefore

$$\angle(m_t, \nu_{\mathcal{H}}) = \angle(m_0, \nu_{\mathcal{H}}) + \frac{L_t^0 - L_t^\pi}{2},$$

which proves the claim in this case (note that before the coupling time ξ only one of the non-decreasing processes L_t^0 and L_t^π is not identically zero).

Since for $t \geq \xi$ we have $Y_t = \left(X_t^1, \ldots, \left|X_t^d\right|\right)$, we have $\angle(m_t, \nu_{\mathcal{H}}) = 0$ which concludes the proof of the claim.

ii) On the time interval $[\sigma_0, \sigma_1)$ the processes are coupled, so there is nothing to prove in this case.

On the time interval $[\sigma_1, \sigma_2)$, in an appropriately chosen coordinate system we have $Y_t = \left(X_t^1, \ldots, \left|X_t^d\right|\right)$, thus the mirror M_t coincides with the boundary hyperplane $\mathcal{H}_d = \left\{\left(z^1, \ldots, z^d\right) \in \mathbb{R}^d : z^d = 0\right\}$ of D_2 where the reflection takes place, thus $M_t \cap D_2 = \varnothing$ in this case.

Inductively, assume the claim is true for $t < \sigma_n$. By continuity, $M_{\sigma_n} \cap D_2 = \varnothing$, thus D_2 lies on one side of M_{σ_n}. By the previous proof, the angle $\angle(m_t, \nu_{\mathcal{H}})$ between m_t and the inward unit normal $\nu_{\mathcal{H}}$ to bounding hyperplane $\mathcal{H}$ of D_2 where the reflection takes place decreases to zero; since D_2 is a convex domain, it follows that on the time interval $[\sigma_n, \sigma_{n+1})$ we have $M_t \cap D_2 = \varnothing$, concluding the proof.

iii) The first part of the claim follows from the previous proof (when the processes are not coupled, the mirror (hence X_t) lies outside D_2; by continuity, it follows that at the coupling time ξ we must have $X_\xi = Y_\xi \in \partial D_2$).

To prove the second part of the claim, consider an arbitrary time interval $[\sigma_n, \sigma_{n+1})$ between two successive hits of Y_t to different bounding hyperplanes of D_2. In an appropriately chosen coordinate system, Y_t is given by Lemma 3.8. After the coupling time ξ, Y_t is given by $Y_t = \left(X_t^1, \ldots, \left|X_t^d\right|\right)$, and therefore if $X_t \in D_2$ (thus $X_t^d \geq 0$) we have $Y_t = \left(X_t^1, \ldots, X_t^d\right) = X_t$, concluding the proof.

iv) Let M_t^α and M_t^β denote the mirrors of the coupling in D^α, respectively D^β, with the same driving Brownian motion X_t.

Since Y_t^α and X_t are symmetric with respect to M_t^α, and Y_t^β and X_t are symmetric with respect to M_t^β, it follows that Y_t^β is obtained from Y_t^β by a rotation which leaves invariant the hyperplane $M_t^\alpha \cap M_t^\beta$, or by a translation by a vector orthogonal to M_t^α (in the case when M_t^α and M_t^β are parallel).

The angle of rotation (respectively the vector of translation) is altered only when either Y_t^α or Y_t^β are on the boundary of D_α, respectively D_β. Since $D_\alpha \subset D_\beta$ are convex domains, the angle of rotation (respectively the vector of translation) decreases when $Y_t^\beta \in D_\beta$ or when $Y_t^\alpha \in \partial D_\alpha$ and $\left(Y_t^\beta - Y_t^\alpha\right) \cdot \nu_{D_\alpha}\left(Y_t^\alpha\right) > 0$ (in these cases Y_t^β and Y_t^α receive a push such that the distance $\left|Y_t^\alpha - Y_t^\beta\right|$ is decreased), thus the maximum distance $\left|Y_t^\alpha - Y_t^\beta\right|$ is attained when $Y_t^\alpha \in \partial D_\alpha$ and $\left(Y_t^\beta - Y_t^\alpha\right) \cdot \nu_{D_\alpha}\left(Y_t^\alpha\right) \leq 0$, and the formula follows. □

4. The proof of Theorem 3.1

By Remark 3.4, it suffices to consider the case when $D_1 = \mathbb{R}^d$ and $D_2 \subset \mathbb{R}^d$ is a convex bounded domain with smooth boundary. To simplify the notation, we will drop the index and write D for D_2 in the sequel.

Let $(D_k)_{k\geq 1}$ be an increasing sequence of convex polygonal domains in $\mathbb{R}^d$ with $\overline{D_n} \subset D_{n+1}$ and $\cup_{n\geq 1} D_n = D$.

Consider $(Y_t^n, X_t)_{t\geq 0}$ a sequence of mirror couplings in $\left(D_n, \mathbb{R}^d\right)$ with starting point $x \in D_1$, with driving Brownian motion $(W_t)_{t\geq 0}$, $W_0 = 0$ given by Theorem 3.9.

By Proposition 3.10, for any $t > 0$ we have

$$\sup_{s\leq t} |Y_s^m - Y_s^n| \leq \operatorname{Dist}(D_n, D_m) = \max_{\substack{x_n \in \partial D_n, x_m \in \partial D_m \\ (x_m - x_n)\cdot \nu_{D_n}(x_n) \leq 0}} |x_n - x_m| \underset{n,m\to\infty}{\to} 0,$$

hence Y_t^n converges a.s. in the uniform topology to a continuous process Y_t.

Since $(Y^n)_{n\geq 1}$ are reflecting Brownian motions in $(D_n)_{n\geq 1}$ and $D_n \nearrow D$, the law of Y_t is that of a reflecting Brownian motion in D, that Y_t is a reflecting Brownian motion in D starting at $x \in D$ (see [8]). Also note that since Y_t^n are adapted to the filtration $\mathcal{F}^W = (\mathcal{F}_t)_{t\geq 0}$ generated by the Brownian motion W_t, so is Y_t.

By construction, the driving Brownian motion Z_t^n of Y_t^n satisfies

$$Z_t^n = \int_0^t G\left(Y_t^n - X_t\right) dW_t, \qquad t \geq 0.$$

Consider the process

$$Z_t = \int_0^t G\left(Y_t - X_t\right) dB_t,$$

and note that since Y is $\mathcal{F}^W$-adapted and $||G|| = 1$, by Lévy's characterization of Brownian motion, Z_t is a free d-dimensional Brownian motion starting at $Z_0 = 0$, also adapted to the filtration $\mathcal{F}^W$.

We will show that Z is the driving process of the reflecting Brownian motion Y_t, i.e. we have

$$Y_t = x + Z_t + L_t^Y = x + \int_0^t G\left(Y_s - B_s\right) dW_s + L_t^Y, \qquad t \geq 0.$$

Note that the mapping $z \longmapsto G(z)$ is continuous with respect to the norm $||A|| = ||(a_{ij})|| = \sum_{i,j=1}^d a_{ij}^2$ of $d \times d$ matrices at all points $z \in \mathbb{R}^d - \{0\}$, hence $G\left(Y_s^n - X_s\right) \underset{n\to\infty}{\to} G\left(Y_s - X_s\right)$ if $Y_s - X_s \neq 0$. If $Y_s - X_s = 0$, then either $Y_s = B_s \in D$ or $Y_s = X_s \in \partial D$.

If $Y_s = B_s \in D$, since $D_n \nearrow D$, there exists $N \geq 1$ such that $B_s \in D_N$, hence $B_s \in D_n$ for all $n \geq N$. By Proposition 3.10, it follows that $Y_s^n = B_s$ for all $n \geq N$, hence in this case we also have $G\left(Y_s^n - B_s\right) = G(0) \underset{n\to\infty}{\to} G(0) = G\left(Y_s - B_s\right)$.

If $Y_s = B_s \in \partial D$, since $\overline{D_n} \subset D$ we have $Y_s^n - B_s \neq 0$, and therefore by the definition of G we have:

$$\begin{aligned}
& \int_0^t \left|\left|G\left(Y_s^n - X_s\right) - G\left(Y_s - X_s\right)\right|\right|^2 1_{Y_s = B_s \in \partial D} ds \\
= & \int_0^t \left|\left|H\left(\frac{Y_s^n - X_s}{||Y_s^n - X_s||}\right) - I\right|\right|^2 1_{Y_s = X_s \in \partial D} ds \\
= & \int_0^t \left|\left|I - 2\frac{Y_s^n - X_s}{||Y_s^n - X_s||}\left(\frac{Y_s^n - X_s}{||Y_s^n - X_s||}\right)' - I\right|\right|^2 1_{Y_s = X_s \in \partial D} ds \\
= & \int_0^t \left|\left|2\frac{Y_s^n - X_s}{||Y_s^n - X_s||}\left(\frac{Y_s^n - X_s}{||Y_s^n - X_s||}\right)'\right|\right|^2 1_{Y_s = X_s \in \partial D} ds \\
= & \; 4\int_0^t 1_{Y_s = X_s \in \partial D} ds \\
\leq & \; 4\int_0^t 1_{\partial D}\left(Y_s\right) ds \\
= & \; 0,
\end{aligned}$$

since Y_t is a reflecting Brownian motion in D, and therefore it spends zero Lebesgue time on the boundary of D.

Since $||G|| = 1$, using the above and the bounded convergence theorem we obtain

$$\lim_{n\to\infty} \int_0^t \left|\left|G\left(Y_s^n - X_s\right) - G\left(Y_s - X_s\right)\right|\right|^2 ds = 0,$$

and therefore by Doob's inequality it follows that

$$E\sup_{s\leq t}|Z_s^n - Z_s|^2 \leq cE\,|Z_t^n - Z_t|^2 \leq cE\int_0^t ||G(Y_s^n - X_s) - G(Y_s - X_s)||^2\,ds \underset{n\to\infty}{\to} 0,$$

for any $t \geq 0$, which shows that Z_t^n converges uniformly on compact sets to $Z_t = \int_0^t G(Y_s - X_s)\,dW_s$.

From the construction, Z_t^n is the driving Brownian motion for Y_t^n, that is

$$Y_t^n = x + Z_t^n + \int_0^t \nu_{D_n}(Y_s^n)\,dL_s^{Y_n},$$

and passing to the limit with $n \to \infty$ we obtain

$$Y_t = x + Z_t + A_t = x + \int_0^t G(Y_s - X_s)\,dW_s + A_t, \qquad t \geq 0,$$

where $A_t = \lim_{n\to\infty}\int_0^t \nu_{D_n}(Y_s^n)\,dL_s^{Y_n}$.

It remains to show that A_t is a process of bounded variation. For an arbitrary partition $0 = t_0 < t_1 < \ldots t_l = t$ of $[0,t]$ we have

$$\begin{aligned} E\sum_{i=1}^{l}\left|A_{t_i} - A_{t_{i-1}}\right| &= \lim_{n\to\infty} E\sum_{i=1}^{l}\left|\int_{t_{i-1}}^{t_i}\nu_{D_n}(Y_s^n)\,dL_s^{Y_n}\right| \\ &\leq \limsup E\; L_t^{Y_n} \\ &= \limsup \int_0^t\int_{\partial D_n} p_{D_n}(s,x,y)\,\sigma_n(dy)\,ds \\ &\leq c\sqrt{t}, \end{aligned}$$

where σ_n is the surface measure on ∂D_n and the last inequality above follows from the estimates in [5] on the Neumann heat kernels $p_{D_n}(t,x,y)$ (see the remarks preceding Theorem 2.1 and the proof of Theorem 2.4 in [7]).

From the above it follows that $A_t = Y_t - x - Z_t$ is a continuous, $\mathcal{F}^W$-adapted process (Y_t, Z_t are continuous, $\mathcal{F}^W$-adapted processes) of bounded variation.

By the uniqueness in the Doob-Meyer semimartingale decomposition of Y_t - reflecting Brownian motion in D, it follows that

$$A_t = \int_0^t \nu_D(Y_s)\,dL_s^Y, \qquad t \geq 0,$$

where L^Y is the local time of Y on the boundary ∂D, and therefore the reflecting Brownian motion Y_t in D constructed above is a strong solution to

$$Y_t = x + \int_0^t G(Y_s - X_s)\,dW_s + \int_0^t \nu_D(Y_s)\,dL_s^Y, \qquad t \geq 0,$$

or equivalent, the driving Brownian motion $Z_t = \int_0^t G(Y_s - X_s)\,dW_s$ of Y_t is a strong solution to

$$Z_t = \int_0^t G\left(\tilde{\Gamma}(y+Z)_s - X_s\right)dW_s, \qquad t \geq 0,$$

concluding the proof of Theorem 3.1.

5. Extensions and applications

As an application of the construction of mirror coupling, we will present a unifying proof of the two most important results on Chavel's conjecture.

It is not difficult to prove that the Dirichlet heat kernel is an increasing function with respect to the domain. Since for the Neumann heat kernel $p_D(t,x,y)$ of a smooth bounded domain $D \subset \mathbb{R}^d$ we have

$$\lim_{t\to\infty} p_D(t,x,y) = \frac{1}{\mathrm{vol}(D)},$$

the monotonicity in the case of the Neumann heat kernel should be reversed.

The above observation was conjectured by Isaac Chavel ([12]), as follows:

Conjecture 5.1 (Chavel's conjecture, [12])**.** *Let $D_{1,2} \subset \mathbb{R}^d$ be smooth bounded convex domains in $\mathbb{R}^d$, $d \geq 1$, and let $p_{D_1}(t,x,y)$, $p_{D_2}(t,x,y)$ denote the Neumann heat kernels in D_1, respectively D_2. If $D_2 \subset D_1$, then*

$$p_{D_1}(t,x,y) \leq p_{D_2}(t,x,y), \tag{5.1}$$

for any $t \geq 0$ and $x,y \in D_1$.

Remark 5.2. The smoothness assumption in the above conjecture is meant to insure the a.e. existence the inward unit normal to the boundaries of D_1 and D_2, that is the boundary should have locally a differentiable parametrization. Requiring that the boundary of the domain is of class $C^{1,\alpha}$ $(0<\alpha<1)$ is a convenient hypothesis on the smoothness of the domains $D_{1,2}$.

In order to simplify the proof, we will assume that $D_{1,2}$ are smooth C^2 domains (the proof can be extended to a more general setup, by approximating $D_{1,2}$ by less smooth domains).

Among the positive results on Chavel conjecture, the most general known results (and perhaps the easiest to use in practice) are due to I. Chavel and W. Kendall (see [12], [16]), and they show that if there exists a ball B centered at either x or y such that $D_2 \subset B \subset D_1$, then the inequality (5.1) in Chavel's conjecture holds true for any $t > 0$.

While there are also other positive results which suggest that Chavel's conjecture is true (see for example [11], [14]), in [4] R. Bass and K. Burdzy showed that Chavel's conjecture does not hold in its full generality (i.e. without additional hypotheses).

We note that both the proof of Chavel (the case when D_1 is a ball centered at either x or y) and Kendall (the case when D_2 is a ball centered at either x or y) relies in an essential way that one of the domains is a ball: the first uses an integration by parts technique, while the later uses a coupling argument of the radial parts of Brownian motion, and none of them can be applied to the other case.

Using the mirror coupling, we can derive a simple, unifying proof of these two important results, as follows:

Theorem 5.3. *Let $D_2 \subset D_1 \subset \mathbb{R}^d$ be smooth bounded domains and assume that D_2 is convex. If for $x,y \in D_2$ there exists a ball B centered at either x or y such that $D_2 \subset B \subset D_1$, then for all $t \geq 0$ we have*

$$p_{D_1}(t,x,y) \leq p_{D_2}(t,x,y). \tag{5.2}$$

Proof. Consider $x,y \in D_2$ fixed and assume without loss of generality that $D_2 \subset B = B(y,R) \subset D_1$ for some $R > 0$.

Consider a mirror coupling (X_t, Y_t) of reflecting Brownian motions in (D_1, D_2) starting at $y \in D_2$.

The idea of the proof is to show that at all times Y_t is at a distance from y smaller than (or equal) to that of X_t from y.

To prove the claim, consider a time $t_0 \geq 0$ when the processes are at the same distance from y, that is $|Y_{t_0} - y| = |X_{t_0} - x|$. If $X_{t_0} = Y_{t_0}$, for $t \geq t_0$ the distances from X_t and Y_t to y will remain equal until the time t_1 when the processes hit the boundary of D_2, and Y_t receives a push in the direction of the inward unit normal to the boundary of D_2. Since D_2 is convex, this decreases the distance of Y_t from y, and the claim follows in this case.

If the processes are decoupled and $|Y_{t_0} - y| = |X_{t_0} - x|$, the hyperplane M_{t_0} of symmetry between X_{t_0} and Y_{t_0} passes through y, and the ball condition shows that we cannot have $X_{t_0} \in \partial D_1$. Therefore for $t \geq t_0$, the processes X_t and Y_t will remain at the same distance from y until $Y_t \in \partial D_2$, when the distance of Y_t from y is again decreased by the local push received as in the previous case, concluding the proof of the claim.

Therefore, for any $\varepsilon > 0$ we have

$$P^x \left(|X_t - y| < \varepsilon \right) \leq P^x \left(|Y_t - y| < \varepsilon \right),$$

and dividing by the volume of the ball $B(y, \varepsilon)$ and passing to the limit with $\varepsilon \searrow 0$, from the continuity of the transition density of the reflecting Brownian motion in the space variable we obtain

$$p_{D_1}(t, x, y) \leq p_{D_2}(t, x, y),$$

for any $t \geq 0$, concluding the proof of the theorem. □

Remark 5.4. As also pointed out by Kendall in [16] (the case when D_2 is a ball), we note that the convexity of the larger domain D_1 is not needed in the above proof in order to derive the validity of condition (5.1) in Chavel's conjecture.

Remark 5.5. We also note that the above proof uses only geometric considerations on the relative position of the reflecting Brownian motions coupled by mirror coupling. Analytically, the above proof reduces to showing that $R_t = |X_{\alpha_t}|^2 - |Y_{\alpha_t}|^2 \geq 0$ for all $0 \leq t < \xi = \inf\{s > 0 : X_s = Y_s\}$, where R_t is the solution of the following stochastic differential equation

$$R_t = R_0 + 2\int_0^t R_s dB_s + 2S_t, \tag{5.3}$$

where $B_t = \int_0^{\alpha_t} \frac{X_s - Y_s}{|X_s - Y_s|^2} \cdot dW_s$ is 1-dimensional Brownian motion, $\alpha_t = A_t^{-1}$ is the inverse of the non-decreasing process A_t defined by

$$A_t = \int_0^t \frac{1}{|X_s - Y_s|^2} ds,$$

and

$$S_t = \int_0^t X_{\alpha_s} \cdot \nu_{D_1}(X_{\alpha_s})\, dL^X_{\alpha_s} - \int_0^t Y_{\alpha_s} \cdot \nu_{D_2}(Y_{\alpha_s})\, dL^Y_{\alpha_s}.$$

Perhaps a better understanding of the mirror coupling, based on the analysis of the local times L^X and L^Y spent by X_t and Y_t on the boundaries of D_1, respectively D_2, in connection to the geometry of the boundaries ∂D_1 and ∂D_2 could give a

proof of Chavel's conjecture for some new classes of convex domains, but so far we were unable to implement it.

We have chosen to carry out the construction of the mirror coupling in the case of smooth domains with $\overline{D_2} \subset D_1$ and D_2 convex, having in mind the application to Chavel's conjecture. However, although the technical details can be considerably longer, it is possible to construct the mirror coupling in a more general setup.

For example, in the case when D_1 and D_2 are disjoint domains, none of the difficulties encountered in the construction of the mirror coupling occur (the possibility of coupling/decoupling), so the constructions extends immediately to this case.

The two key ingredients in our construction of the mirror coupling were the hypothesis $\overline{D_2} \subset D_1$ (needed in order to reduce by a localization argument the construction to the case $D_1 = \mathbb{R}^d$) and the hypothesis on the convexity of the inner domain D_2 (which allowed us to construct a solution of the equation of the mirror coupling in the case $D_1 = \mathbb{R}^d$).

Replacing the first hypothesis by the condition that the boundaries ∂D_1 and ∂D_2 are not tangential (needed for the localization of the construction of the mirror coupling) and the second one by condition that $D_1 \cap D_2$ is a convex domain, the arguments in the present construction can be modified in order to give rise to a mirror coupling of reflecting Brownian motion in (D_1, D_2).

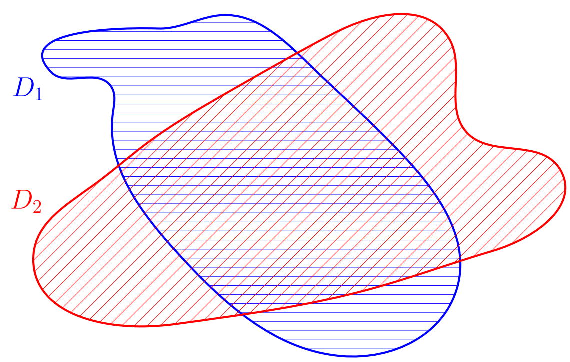

FIGURE 3. Generic smooth domains $D_{1,2} \subset \mathbb{R}^d$ for the mirror coupling: D_1, D_2 have non-tangential boundaries and $D_1 \cap D_2$ is a convex domain.

We conclude with some remarks on the non-uniqueness of the mirror coupling in general domains. To simplify the ideas, we will restrict to the 1-dimensional case when $D_2 = (0, \infty) \subset D_1 = \mathbb{R}$.

Fixing $x \in (0, \infty)$ as starting point of the mirror coupling (X_t, Y_t) in (D_1, D_2), the equations of the mirror coupling are

$$X_t = x + W_t \tag{5.4}$$

$$Y_t = x + Z_t + L_t^Y \tag{5.5}$$

$$Z_t = \int_0^t G(Y_s - X_s)\, dW_s \tag{5.6}$$

where in this case

$$G(z) = \begin{cases} -1, & \text{if } z \neq 0 \\ +1, & \text{if } z = 0 \end{cases}.$$

Until the hitting time $\tau = \{s > 0 : Y_s \in \partial D_2\}$ of the boundary of ∂D_2 we have $L_t^Y \equiv 0$, and with the substitution $U_t = -\frac{1}{2}(Y_t - X_t)$, the stochastic differential for Y_t becomes

$$U_t = \int_0^t \frac{1 - G(Y_s - X_s)}{2} dW_s = \int_0^t \sigma(U_s)\, dW_s, \tag{5.7}$$

where

$$\sigma(z) = \frac{1 - G(z)}{2} = \begin{cases} 1, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}.$$

By a result of Engelbert and Schmidt ([13]) the solution of the above problem is not even weakly unique, for in this case the set of zeroes of the function σ is $N = \{0\}$ and σ^{-2} is locally integrable on $\mathbb{R}$.

In fact, more can be said about the solutions of (5.7) in this case. It is immediate that both $U_t \equiv 0$ and $U_t = W_t$ are solutions to 5.7, and it can be shown that an arbitrary solution can be obtained from W_t by delaying it when it reaches the origin (sticky Brownian motion with sticky point the origin).

Therefore, until the hitting time τ of the boundary, we obtain as solutions

$$Y_t = X_t = x + W_t \tag{5.8}$$

and

$$Y_t = X_t - 2W_t = x - W_t, \tag{5.9}$$

and an intermediate range of solutions, which agree with (5.8) for some time, then switch to (5.9) (see [18]).

Correspondingly, this gives rise to mirror couplings of reflecting Brownian motions for which the solutions stick to each other after they have coupled (as in (5.8)), or they immediately split apart after coupling (as in (5.9)), and there is a whole range of intermediate possibilities. The first case can be referred to as *sticky* mirror coupling, the second as *non-sticky* mirror coupling, and the intermediate possibilities as *weak/mild sticky* mirror coupling.

The same situation occurs in the general setup in $\mathbb{R}^d$, and it is the cause of lack uniqueness of the stochastic differential equations which define the mirror coupling. In the present paper we detailed the construction of the sticky mirror coupling, which we considered to be the most interesting, both from the point of view of constructions and of the applications, although the other types of mirror coupling might prove useful in other applications.

Acknowledgement

I would like to thank Krzysztof Burdzy and Wilfried S. Kendall for the helpful discussions and the encouragement to undertake the task of the present project.

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